International Journal of Theoretical Physics, Vol. 22, No. 11, 1983

Homotopy Construction of a Spinor Wave Functional

P. L. Antonelli

Department of Mathematics, University of Alberta, Edmonton, Alberta, T6G 2G1 Canada

and J. G. Williams

Department of Mathematics, Okanagan College, P.O. Box 1088, Vernon, British Columbia, V1T 6N2 Canada

Received January 26, 1981

A nonlinear field theory is studied for which the field variables range over a 3-sphere. The Whitehead integral formula is used to construct a double-valued spinor as a functional of the original single-valued field variables.

1. INTRODUCTION

Although the construction of scalar fields from spinors is relatively trivial, it is not obvious how to do the reverse. Nevertheless, such a construction is sometimes possible. Given a field theory that admits kinks and half-odd-integer spin (in the topological sense), the existence of double-valued spinor functionals of the original (single-valued) field variables has been established for some time (Enz, 1963; Finkelstein and Misner, 1959; Finkelstein, 1966; Skyrme, 1958, 1961a, 1962). It is the purpose of this paper to construct such a wave functional $\Psi(\varphi)$ for the case in which the range of the field variables is a 3-sphere, S^3.

A particular field configuration is defined by giving a mapping

$$\varphi : R^3 \to S^3$$

with

$$\varphi(\mathbf{x}) \to y_0 \qquad \text{as } |\mathbf{x}| \to \infty$$

where $\mathbf{x} \in R^3$, and $y_0 \in S^3$ is a fixed point. Kinks and half-odd-integer spin are known to be admitted (Williams and Zvengrowski, 1977). In what follows it will be convenient to compactify R^3, replacing it by a 3-sphere. Assume that this is done in some standard fashion, for example, by using the stereographic projection. Henceforth, φ will be assumed to map a 3-sphere into a 3-sphere. For emphasis, we shall sometimes write S_D^3 and S_R^3 for the domain 3-sphere and the range 3-sphere, respectively. Introduce homogeneous coordinates $(\phi_1, \phi_2, \phi_3, \phi_4) \equiv (\boldsymbol{\phi}, \phi_4) \in S_R^3$ and $(\psi_1, \psi_2, \psi_3, \psi_4) \equiv (\boldsymbol{\psi}, \psi_4) \in S_D^3$. Let $(0,0,0,1) \in S_R^3$ and $(0,0,0,1) \in S_D^3$ be fixed points. We shall consider mappings

$$\varphi : S_D^3 \to S_R^3$$

with

$$\varphi(0,0,0,1) = (0,0,0,1)$$

To specify a particular mapping φ, the functional form $(\boldsymbol{\phi}, \phi_4) = \varphi(\boldsymbol{\psi}, \psi_4)$ must be given. A simple example would be the constant mapping φ_0 given by

$$\boldsymbol{\phi} = \mathbf{0}, \qquad \phi_4 = 1$$

An equally simple example would be the identity mapping φ_I given by

$$\phi_\mu = \psi_\mu, \qquad \mu = 1,2,3,4$$

Instead of parametrizing S_D^3 with homogeneous coordinates, we may use angular coordinates θ, ϕ, ψ. The relationship to $(\boldsymbol{\psi}, \psi_4)$ is given by

$$\psi_1 = \sin\frac{\theta}{2} \cdot \sin\left(\frac{\psi - \phi}{2}\right)$$

$$\psi_2 = \sin\frac{\theta}{2} \cdot \cos\left(\frac{\psi - \phi}{2}\right)$$

$$\psi_3 = \cos\frac{\theta}{2} \cdot \sin\left(\frac{\psi + \phi}{2}\right)$$

$$\psi_4 = \cos\frac{\theta}{2} \cdot \cos\left(\frac{\psi + \phi}{2}\right) \tag{1}$$

The range sphere S_R^3 can be simularly parametrized by angular coordinates α, β, γ. Sacrificing rigor for clarity, mappings between these sets of angular variables will also be denoted by φ, writing $(\alpha, \beta, \gamma) = \varphi(\theta, \phi, \psi)$. Thus $\varphi_0(\theta, \phi, \psi) = (0,0,0)$ and $\varphi_I(\theta, \phi, \psi) = (\theta, \phi, \psi)$.

It will be necessary to parametrize the rotation group, $SO(3)$. This can be done (locally) by giving a unit vector $\boldsymbol{\omega}$ to indicate the axis of rotation and an angle ψ. This choice of notation, i.e., the use of ψ in two different contexts, is for later convenience. Vectors $\mathbf{x} \in R^3$ transform under rotation according to

$$x_i' = \sum R_{ij}(\psi) x_j$$

where

$$R_{ij}(\psi) = \delta_{ij} \cos\psi + \omega_i \omega_j (1 - \cos\psi) + \sum \varepsilon_{ijk} \omega_k \sin\psi \tag{2}$$

As ψ varies between 0 and 2π then $R_{ij}(\psi)$ represents a 2π-rotation loop in $SO(3)$. This is known to be nontrivial (in the sense that it is not deformable to a point) and belongs to the generating class of $\pi_1(SO(3)) \approx Z_2$. It has been agreed to compactify R^3 with the stereographic projection

$$\psi_i = 2ax_i/(r^2 + a^2), \qquad i = 1,2,3$$

$$\psi_4 = (r^2 - a^2)/(r^2 + a^2)$$

where $r = |\mathbf{x}|$ and a is a constant. Thus points of S_D^3 transform according to

$$\psi_i' = \sum R_{ij}(\psi) \psi_j, \qquad i = 1,2,3$$

$$\psi_4' = \psi_4$$

It is now possible to define rotations in mapping space by

$$\varphi' = R(\psi)\varphi$$

where

$$[R(\psi)\varphi](\{\psi_i\}, \psi_4) = \varphi\left(\{\sum R_{ij}^{-1}(\psi)\psi_j\}, \psi_4\right)$$

It is the object of this paper to construct a wave functional $\Psi(\varphi)$ which is double-valued under 2π rotation of the mapping space, in the sense that $\Psi(R(2\pi)\varphi) \neq \Psi(\varphi)$. [It is, of course, true that $R(2\pi)\varphi = \varphi$).] More specifically, the construction will be such as to obtain the sign flip usual for a spinor:

$$\Psi(R(2\pi)\varphi) = -\Psi(\varphi)$$

$$\Psi(R(4\pi)\varphi) = \Psi(\varphi) \tag{3}$$

Note that the quantum mechanics in this paper is assumed to be in the "field representation," i.e., in the Schrödinger picture. This representation was popular in the early years of quantum field theory (although it leads to difficulties with zero-point energies). Recent papers from this point of view include Marx (1969), Kimstedt (1979), and Ragiadakos (1980).

2. INTEGRAL FORMULAS

Since the mechanism by which half-odd-integer spin arises in kink theories is topological, consider some functionals which are relevant to topology. One of these is obtained by integrating the Jacobian of the mapping $(\alpha, \beta, \gamma) = \varphi(\theta, \phi, \psi)$:

$$f(\varphi) = (4\pi^3)^{-1} \iiint_{S_D^3} \frac{\partial(\alpha, \beta, \gamma)}{\partial(\theta, \phi, \psi)} d\theta\, d\phi\, d\psi \tag{4}$$

The functional $f(\varphi)$ takes integer values $n \in Z$, where n is the degree of mapping, deg φ. In the present context, n is called the *kink number* of φ. This is related to the idea of homotopy class. Since $\pi_3(S^3) \approx Z$, mappings between 3-spheres can be labeled by the integers. Thus the set of homotopy classes can be written: $\ldots, Q_{-2}, Q_{-1}, Q_0, Q_1, Q_2, \ldots \in \pi_3(S^3)$. For example, $\varphi_0 \in Q_0$ and $\varphi_1 \in Q_1$. Degree and homotopy class label are synonymous in this case, since a theorem due to Hopf (Milnor, 1972, p. 51) states: If M^m is a connected, orientable, and boundaryless m-manifold then two maps $M^m \to S^m$ are homotopic if and only if they have the same degree. (It is important that the range should be a sphere. Given mappings $f, g: S^1 \times S^2 \to RP^3$, for example, deg $f =$ deg g is a necessary but not a sufficient condition for f and g to be homotopic.) Equation (4) can be generalized to the higher-dimensional case, $S^m \to S^m$, in an obvious manner (Patani, Schlindwein, and Shafi, 1976).

Functionals defined as integrals of Jacobians only apply when mapping between manifolds of the same dimension. As will be explained in Section 3, spin properties are controlled by mappings $S^4 \to S^3$. Integrals akin to Eq. (4) but relevant to mappings $S^m \to S^n$, $m \neq n$, are very rare. Indeed, the whole subject of algebraic topology was invented to avoid dealing with such analytic expressions. However, mappings $\chi: S^3 \to S^2$ can be characterized by the Whitehead integral (Whitehead, 1947) which has found recent application in physics (De Ritis, Finkelstein, Pisello, and Weil, 1978; De Vega, 1978; Minami, 1979, 1980; Nicole, 1978; Pisello, 1977, 1978, 1979; Ryder, 1980; Williams, 1979; Woo, 1977).

Let Θ, Φ be local coordinates on S^2 and let (θ, ϕ, ψ) be local coordinates on S^3. The function χ can be expressed locally as $(\Theta, \Phi) = \chi(\theta, \phi, \psi)$. Let σ be an area density on S^2. [It would be most natural to take

$\sigma = (4\pi)^{-1}\sin\Theta$.] Define

$$F_{ij} = \sigma \cdot \left(\partial_i \Theta \partial_j \Phi - \partial_j \Theta \partial_i \Phi \right)$$

where the indices i and j refer to any of θ, ϕ, ψ. It can be checked that

$$\partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} = 0$$

and hence by de Rham's theorem there is a (nonunique) vector field $\mathbf{A}$ defined over the whole of S^3 such that

$$\partial_i A_j - \partial_j A_i = F_{ij}$$

The Whitehead integral is defined by

$$H(\chi) = \iiint_{S^3} \sigma \cdot \begin{vmatrix} A_1 & A_2 & A_3 \\ \dfrac{\partial\Theta}{\partial\theta} & \dfrac{\partial\Theta}{\partial\phi} & \dfrac{\partial\Theta}{\partial\psi} \\ \dfrac{\partial\Phi}{\partial\theta} & \dfrac{\partial\Phi}{\partial\phi} & \dfrac{\partial\Phi}{\partial\psi} \end{vmatrix} d\theta\, d\phi\, d\psi$$

The functional $H(\chi)$, known as the Hopf invariant of χ, takes values in the integers. Any two mappings $S^3 \to S^2$ are homotopic if and only if they have the same Hopf invariant (Hilton, 1966, p. 71, 74). We note that $\pi_3(S^2) \approx Z$.

The Whitehead integral can be expressed more concisely using differential forms (Whitehead, 1947; Flanders, 1963, p. 79):

$$H(\chi) = \iiint_{S^3} \alpha_1 \wedge \chi^* \sigma_2 \tag{5}$$

where σ_2 is an area two-form on S^2, $\chi^*\sigma_2$ denotes the corresponding pulled-back two-form on S^3, and α_1 is a one-form on S^3 with the property that $d\alpha_1 = \chi^*\sigma_2$. The one-form α_1 is globally defined on S^3 and is unique only up to the differential of a function.

The Hopf map $h: S^3 \to S^2$ (Hilton, 1966) is an example of a mapping of Hopf invariant 1. There are several versions of h and, to be explicit, we shall follow the convention of Minami (1979). Parametrize S^2 by homogeneous variables ξ_i, $i = 1, 2, 3$ with $\Sigma \xi_i^2 = 1$. The Hopf map can then be defined by $(\xi_1, \xi_2, \xi_3) = h(\psi_1, \psi_2, \psi_3, \psi_4)$ where

$$\begin{aligned} \xi_1 &= 2(\psi_1\psi_3 + \psi_2\psi_4) \\ \xi_2 &= 2(\psi_2\psi_3 - \psi_1\psi_4) \\ \xi_3 &= 1 - 2(\psi_1^2 + \psi_2^2) \end{aligned} \tag{6}$$

Alternatively, S^2 can be parametrized by a polar angle Θ and an azimuthal angle Φ according to

$$\xi_1 = \sin\Theta\cos\Phi$$

$$\xi_2 = \sin\Theta\sin\Phi$$

$$\xi_3 = \cos\Theta$$

Using Eq. (1) to express the $\{\psi_i\}$ in terms of the angular variables (θ, ϕ, ψ), the Hopf map h is given by $(\Theta, \Phi) = h(\theta, \phi, \psi)$ where

$$\Theta = \theta, \qquad \Phi = \phi$$

Note that h is a fiber mapping, and maps great circles in S^3 into single points of S^2. The angle ψ is the angle of the S^1 fiber.

Let us check that $H(h)$ is equal to unity. Choosing $\sigma_2 = (4\pi)^{-1}\sin\Theta\, d\Theta \wedge d\Phi$ we obtain

$$h^*\sigma_2 = (4\pi)^{-1}\sin\theta\, d\theta \wedge d\phi$$

This leads to

$$\alpha_1 = -(4\pi)^{-1}(\cos\theta\, d\phi + d\psi).$$

From Eq. (5),

$$H(h) = -(4\pi)^{-2}\iiint \sin\theta\, d\theta \wedge d\phi \wedge d\psi$$

Interestingly enough, this turns out to equal $-1/2$, instead of one! The negative sign is no problem since it depends upon the orientation convention. Minami (1979) has given an explanation for the factor of $1/2$. One is presumably regarding θ, ϕ, ψ as some sort of Euler angles varying according to $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \phi$, $\psi \leqslant 2\pi$. However, such angles are known to cover only $SO(3) = S^3/Z_2$, so that only half of S^3 is being covered. One possibility is to decouple the ϕ and ψ by taking the combinations $(\psi - \phi)/2$ and $(\psi + \phi)/2$. This then leads to a Hopf invariant of unit magnitude. In this present paper we shall keep the same θ, ϕ, ψ but take care to always interpret them not as Euler angles but as variables which range over the whole of S^3.

Kervaire (1953) has generalized Eq. (5) to

$$H_{\text{gen}}(\chi) = \int_{S^{2n-1}} \alpha_{n-1} \wedge \chi^*\sigma_n \tag{7}$$

for mappings $\chi: S^{2n-1} \to S^n$. This generalized Hopf invariant is zero for all χ, unless $n = 2$, 4, or 8 (Husemoller, 1974, Theorem 4.3). Unfortunately, Eq. (7) does not include the $S^4 \to S^3$ case, which is of interest for the study of spin. Since $\pi_4(S^3) \approx Z_2$, there are only two homotopy classes. As far as we know, there is no integral that can be used directly to classify maps $S^4 \to S^3$, and so we shall use Eq. (5) and relate the Hopf invariant to the $S^4 \to S^3$ case by means of suspension. The relevance of the Hopf map to spin has been noticed previously (Williams, 1970; Hellsten, 1979, p. 2432 and footnote 6; see also Pak and Tze, 1979).

3. $S^4 \to S^3$ MAPPINGS

Consider maps from R^3 (or S^3) into S^3. Since the range S^3 is a topological group, it follows that the mapping space is also a topological group (Spanier, 1966, p. 34). Since the pathwise-connected components of a topological group are homeomorphic (Montgomery and Zippin, 1955, p. 39), the homotopy classes $\ldots, Q_{-2}, Q_{-1}, Q_0, Q_1, Q_2, \ldots$ are all homeomorphic to each other. Let us use the identity map φ_I as the homeomorphism between Q_i and Q_{i+1}, all $i \in Z$. Thus if φ denotes any 1-kink map, $\varphi \in Q_1$, then a corresponding 0-kink map $\varphi_{\text{zero}} \in Q_0$ can be found according to

$$\varphi_{\text{zero}}(\mathbf{x}) = [\varphi_I(\mathbf{x})]^{-1} \cdot \varphi(\mathbf{x}) \tag{8}$$

Note that $[\varphi_I(\mathbf{x})]^{-1}$ denotes the group inverse in S^3. Since we wish Eq. (3) to be satisfied for any 1-kink mapping φ, 2π-rotation paths in Q_1 should be nontrivial. More specifically, $\pi_1(Q_1)$ should have an element of order 2 which contains all the 2π-rotation paths. Given any $\varphi \in Q_1$, Eq. (8) allows a corresponding $\varphi_{\text{zero}} \in Q_0$ to be calculated. Thus it is sufficient to consider paths in Q_0. These will not be 2π-rotation paths. Equation (8) maps (non)trivial paths in Q_1 into (non)trivial paths in Q_0, but it does not map rotation paths in Q_1 into rotation paths in Q_0.

Let p be any path in Q_0 beginning and ending at the constant mapping φ_0:

$$p: R^1 \to Q_0$$

$$p(-\infty) = p(\infty) = \varphi_0$$

Define a mapping

$$\tilde{\varphi}: R^4 \to S^3$$

by

$$\tilde{\varphi}(\mathbf{x}, u) = [p(u)](\mathbf{x})$$

Thus there is a correspondence between paths in Q_0 and mappings from $R^4 \to S^3$. Since $\tilde{\varphi}(x, u) \to (0,0,0,1)$ as either of $|\mathbf{x}|$ or $|u| \to \infty$ it follows that R^4 can be compactified to give mappings $S^4 \to S^3$. Hence

$$\pi_1(Q_1) \approx \pi_1(Q_0) \approx \pi_4(S^3) \approx Z_2$$

and the desired double-valuedness is obtained. That it is due specifically to 2π-rotation was demonstrated by Williams and Zvengrowski (1977). The relevance to spin of the fourth homotopy group, π_4, is well known and was first emphasized by Finkelstein and Misner (1959).

Given any 1-kink mapping $\varphi: S^3 \to S^3$ and a 2π-rotation loop, the above procedure can be used to find a corresponding 0-kink mapping $\varphi_{\text{zero}} \in Q_0$ and a loop (*not* a rotation loop) in Q_0, and hence to find a corresponding mapping. $\tilde{\varphi}: S^4 \to S^3$. From $\tilde{\varphi}$ we wish to find a corresponding map $\varphi^{\#}$ from a 3-sphere, S^3 into a 2-sphere, S^2. The Hopf invariant of $\varphi^{\#}$ will lead to the desired wave functional (which will be a functional of the original φ). The method of constructing this wave functional will be explained in the next section.

4. CONSTRUCTION OF $\Psi(\varphi)$

Given $\varphi: S^3 \to S^3$ and a 2π-rotation loop we obtain $\tilde{\varphi}: S^4 \to S^3$ with $\tilde{\varphi}(0,0,0,0,1) = (0,0,0,1)$. The set $D = \{(\phi, \phi_4)|\phi_3 = 0\}$ is a standard 2-sphere in S^3. From results in Thom's transverse regularity theory we may suppose without loss of generality that $\tilde{\varphi}$ is transverse regular on D. [For a discussion of mappings that are "well behaved" in the sense of being transverse regular, see Guillemin and Golubitsky (1973)]. If $\tilde{\varphi}$ were not so, there is a homotopy of $\tilde{\varphi}$ to a map of S^4 onto S^3 which is transverse regular on D. What is more, the new mapping may also be chosen close to $\tilde{\varphi}$ in the sense of δ approximation (Conner and Floyd, 1964, p. 21). Transversality implies that $\tilde{\varphi}^{-1}(D)$ is a compact boundaryless manifold of dimension 3, smoothly (i.e., C^∞) imbedded in S^4 (Stong, 1968, Appendix 2, p. 23). Denote $\tilde{\varphi}^{-1}(D)$ by M^3 and denote the restriction of $\tilde{\varphi}$ to M^3 by $\varphi^{\#}$. The maximal rank of $\varphi^{\#}$ (in the Jacobian sense) and the fact that the only compact 1-manifold without boundary is S^1, the unit circle, imply that $\varphi^{\#}$ is a locally trivial smooth fiber bundle map with S^1 as fiber. Such bundles have been classified by Steenrod (1951, p. 135). This classification implies that M^3 must be a lens space $L^3(m,1)$ of type $(m,1)$. Here m is a nonnegative integer and M^3 is the smooth orbit space of a free action of Z_m on S^3. It follows that $\pi_1(M^3) \approx Z_m$. It is a fact that lens spaces can occur for every choice of m and that these are topologically distinct (even bundle equivalence distinct!) Two special cases are particularly simple. If $m=0$ then M^3 is homeomorphic to $S^1 \times S^2$, and if $m=1$ then M^3 is homeomorphic to S^3. We shall establish the following proposition.

Proposition. Given that M^3 is a lens space $L^3(m,1)$, then the embedding $M^3 \subseteq S^4$ necessarily means that $m=0$ or $m=1$.

Proof. The argument we are about to give does not appear in the literature as a whole. It is probably due to J. H. C. Whitehead. (We would like to thank C. S. Hoo for this point.)

Assume $m \neq 0,1$. For $m \neq 0,1$, the Z_m-cohomology groups of $M^3 \equiv L^3(m,1)$ are

$$H^i(L^3; Z_m) \approx \begin{cases} Z_m, & i=0,1,2,3 \\ 0, & i \geqslant 4 \end{cases}$$

(Hilton and Wylie, 1967, p. 223 and Theorem 3.9.6, p. 135).

First we show that the removal of M^3 from S^4 will yield exactly two connected components A and B. We need only apply the Alexander duality theorem (Spanier, 1966, Theorem 16, p. 296), so that

$$\tilde{H}_0(S^4 - M^3) \approx \bar{H}^3(M^3) \approx H^3(M^3) \approx Z_m$$

The tilde denotes the reduced homology group and the bar denotes a certain limit group which, because M^3 is tautly imbedded in S^4, is isomorphic to the usual cohomology group $H^3(M^3; Z_m)$. The embedding is *taut* because the pair (S^4, M^3) is a smooth compact pair and is hence a compact polyhedral pair. This means that in a suitable triangulation of S^4, M^3 is a subcomplex. Since $\tilde{H}_0$ is isomorphic to one copy of Z_m, there are exactly *two* connected components in $S^4 - M^3$. We have denoted these by A and B. Taking closures we have $\bar{A} \cup \bar{B} = S^4$ and $\bar{A} \cap \bar{B} = M^3$. A similar argument to the above now yields

$$0 \approx \tilde{H}_0(\bar{B}) \approx \tilde{H}_0(S^4 - \bar{A}) \approx \bar{H}^3(\bar{A}) \approx H^3(\bar{A})$$

Hence $H^3(\bar{A}; Z_m) \approx 0$, and likewise $H^3(\bar{B}; Z_m) \approx 0$. We shall need these isomorphisms shortly.

We now apply the Mayer–Vietoris exact cohomology sequence (Spanier, 1966, Theorem 10, p. 239). Using Z_m coefficients we have

$$\cdots \to H^1(S^4) \xrightarrow{j^*} H^1(\bar{A}) \oplus H^1(\bar{B}) \xrightarrow{i^*} H^1(L) \xrightarrow{\delta^*} H^2(S^4) \to \cdots$$

where the inclusions $j_1 : \bar{A} \to S^4$, $i_1 : L \to \bar{A}$, $j_2 : \bar{B} \to S^4$ and $i_2 : L \to \bar{B}$ are used to define j^* and i^* according to:

$$j^*(u) = (j_1^* u, j_2^* u)$$

$$i^*(a, b) = i_1^*(a) - i_2^*(b)$$

Now since $H^1(S^4) \approx H^2(S^4) \approx 0$, exactness yields the isomorphism

$$i^* : H^1(\bar{A}) \oplus H^1(\bar{B}) \xrightarrow{\approx} H^1(L)$$

Let (a, b) be a generator of $H^1(\bar{A}) \oplus H^1(\bar{B})$. Then $X = i_1^*(a) - i_2^*(b)$ generates $H^1(L) \approx Z_m$. Letting

$$\beta_{(q)} : H^q(L; Z_m) \to H^{q+1}(L; Z_m)$$

denote the Bockstein homomorphism corresponding to the coefficient sequence

$$0 \to Z_m \to G \to Z_m \to 0$$

(Spanier, 1966, p. 269–270) we have the cup product formula

$$X \cup \beta_{(1)} X = i_1^*(a \cup \beta_{(1)} a) - i_2^*(b \cup \beta_{(1)} b)$$

But

$$a \cup \beta_{(1)} a \in H^3(\bar{A}; Z_m) \approx 0$$

and

$$b \cup \beta_{(1)} b \in H^3(\bar{B}; Z_m) \approx 0$$

so that

$$X \cup \beta_{(1)} X = 0$$

On the other hand, the cohomology structure of $L^3(m, 1) = M^3$ has the property that if $X \in H^1(L; Z_m)$ generates, then $\beta_{(1)} X \in H^2(L; Z_m) \approx Z_m$

also generates, and furthermore $X \cup \beta_{(1)} X$ generates $H^3(L; Z_m) \approx Z_2$ (Hilton and Wylie, 1967, p. 225). Hence,

$$X \cup \beta_{(1)} X \neq 0$$

Comparing this with the equation displayed previously, we obtain a contradiction. Hence $m = 0$ or $m = 1$. ■

The above analysis means that starting from a mapping $\varphi: S^3 \to S^3$ we can find $\tilde{\varphi}: S^4 \to S^3$ and that we then have a well defined prescription for constructing a mapping $\varphi^{\#}$ with either $\varphi^{\#}: S^1 \times S^2 \to S^2$ ($m = 0$ case) or $\varphi^{\#}: S^3 \to S^2$ ($m = 1$ case). (Loosely speaking, the prescription consists of picking a diameter $S^2 \subset S^3$ and "working backwards".) Let us further assume that φ is a degree 1 mapping (i.e., there is one kink present). This will rule out the $S^1 \times S^2$ case since the latter can only result from an inessential $\tilde{\varphi}: S^4 \to S^3$ which contradicts our deg $\varphi = 1$ assumption. Thus 1-kink mappings $\varphi: S^3 \to S^3$, together with a 2π-rotation loop, give rise to mappings $\varphi^{\#}: S^3 \to S^2$. The Hopf invariant of $\varphi^{\#}$ can be defined according to the Whitehead integral, Eq. (5). With a slight change of notation, Eq. (5) can be rewritten to display the integral over the angular variable along the fiber:

$$H(\varphi^{\#}) = -(4\pi)^{-1} \iiint_{S^3} \varphi^{\#*}\sigma_2 \wedge d\psi$$

Since $\varphi^{\#}$ depends on φ, $H(\varphi^{\#})$ is a functional of φ. Allowing for a variable upper limit on the ψ integral, we can introduce a corresponding indefinite integral. This will depend on $\varphi^{\#}$ and hence on φ. Thus we define the functional $F(\varphi; \psi)$ according to

$$F(\varphi; \psi) = -(4\pi)^{-1} \iiint_{\psi=0}^{\psi} \varphi^{\#*}\sigma_2 \wedge d\psi \tag{9}$$

The fiber parameter, ψ, depends (modulo 2π) upon the choice of φ. Given any $\varphi: S^3 \to S^3$ we can determine $F(\varphi; \psi)$. Suppressing the ψ, $F(\varphi)$ becomes a many-valued functional. Clearly, the more times that ψ sweeps through the S^1 fiber, the greater will be the value of $F(\varphi)$.

We note that ψ is the angle of rotation, i.e., a rotation operator R_ψ (or R_ψ^{-1}) acting on φ will cause the angle $\psi \in S^1 = SO(2) \subset SO(3)$ to rotate. This follows from the fact that a generator of $\pi_3(S^2) \approx Z$ is in correspondence with an $SO(2)$ rotation loop in the space of 1-kink maps, $S^2 \to S^2$, and that the generator of $\pi_4(S^3) \approx Z_2$ is in correspondence with an $SO(3)$ rotation loop in the space of 1-kink maps, $S^3 \to S^3$, a result previously established by Williams and Zvengrowski (1977). The operator R_ψ denotes a three-dimensional rotation [in accord with Eq. (2)], the unit vector ω denoting the axis of rotation having been suppressed. All three upper limits for the integrals in Eq. (9) could be variable. However, we assume for simplicity of notation that some definite axis of rotation has been selected.

For any integer n, $F(\varphi; 2\pi n) \in Z$. Since we are looking for Z_2 behavior (i.e., double-valuedness as opposed to many-valuedness) we define

$$\Psi(\varphi) = e^{i\pi F(\varphi)} \tag{10}$$

This functional has the desired double-valuedness expressed by Eq. (3).

The familiar notion of a spinor as being "something with components" can be introduced by defining

$$\begin{pmatrix} \Psi_+(\varphi) \\ \Psi_-(\varphi) \end{pmatrix} = \begin{pmatrix} e^{+i\pi F(\varphi)} \\ e^{-i\pi F(\varphi)} \end{pmatrix}$$

It transforms according to the usual transformation rule for spinors using $\exp(i\sigma\cdot\omega\psi/2)$ where the $\{\sigma_i\}$ denote the Pauli matrices. This can easily be checked for rotations about the z axis, $\omega = (0,0,1)$:

$$\begin{pmatrix} \Psi'_+(\varphi) \\ \Psi'_-(\varphi) \end{pmatrix} = R_\psi \begin{pmatrix} \Psi_+(\varphi) \\ \Psi_-(\varphi) \end{pmatrix} = \begin{pmatrix} R_\psi e^{+i\pi F(\varphi)} \\ R_\psi e^{-i\pi F(\varphi)} \end{pmatrix}$$

$$R_\psi e^{\pm i\pi F(\varphi)} = R_\psi \cdot \exp\left\{ \pm i\pi \left[-(4\pi)^{-1} \int\int\int^{\psi_0} \chi^* \sigma_2 \, d\psi_0 \right] \right\}$$

$$= \exp\left\{ \pm i\pi \left[-(4\pi)^{-1} \int\int\int^{\psi_0+\psi} \chi^* \sigma_2 \, d\psi_0 \right] \right\}$$

$$= e^{\pm i\pi\psi/(2\pi)} e^{\pm i\pi F(\varphi)}$$

Thus

$$\begin{pmatrix} \Psi'_+(\varphi) \\ \Psi'_-(\varphi) \end{pmatrix} = e^{i\sigma_3\psi/2} \begin{pmatrix} \Psi_+(\varphi) \\ \Psi_-(\varphi) \end{pmatrix}$$

5. AN EXAMPLE

Take the identity map $\varphi_{\mathrm{I}}: S^3 \to S^3$ as an example of a degree 1 mapping, $\varphi_{\mathrm{I}} \in Q_1$. This is given by $\phi_\mu = \psi_\mu$, $\mu = 1,2,3,4$. The corresponding

2π-rotation path in Q_1 is defined by

$$\phi_i = \sum R_{ij}(\psi)\psi_j, \qquad i=1,2,3$$
$$\phi_4 = \psi_4$$

where the $R_{ij}(\psi)$ are given in terms of the direction ω and the angle ψ in Eq. (2). Since S^3 and $SU(2)$ are homeomorphic, the above 2π-rotation loop in Q_1 can be represented in terms of the Pauli matrices $\sigma_1, \sigma_2, \sigma_3 \in SU(2)$:

$$I\phi_4 + i\boldsymbol{\sigma}\cdot\boldsymbol{\phi} = I\psi_4 + i\sum \sigma_i R_{ij}(\psi)\psi_j$$

The inverse element $I\psi_4 - i\boldsymbol{\sigma}\cdot\boldsymbol{\psi}$ maps from Q_1 to Q_0, and so it follows from Eq. (8) that the corresponding loop in Q_0 is given by

$$I\phi_4 + i\boldsymbol{\sigma}\cdot\boldsymbol{\phi} = (I\psi_4 - i\boldsymbol{\sigma}\cdot\boldsymbol{\psi})\left[I\psi_4 + i\sum \sigma_i R_{ij}(\psi)\psi_j\right] \tag{11}$$

This is equivalent to a mapping $\tilde{\varphi}: S^4 \to S^3$. The angle ψ plays the role of the extra variable. It is clearly immaterial whether ψ varies in $[0, 2\pi]$ or $(-\infty, +\infty)$. To pick out a diameter of S^3 we put $\phi_3 = 0$:

$$0 = \phi_3 = \sum \psi_4 R_{3j}(\psi)\psi_j - \psi_3\psi_4 + \sum \psi_1 R_{2j}(\psi)\psi_j - \sum \psi_2 R_{1j}(\psi)\psi_j$$

This leads to $\omega_1 = \omega_2 = 0$, $\omega_3 = 1$, $\psi = \pi$, i.e., a rotation of π about the z axis [$\psi = 0 \pmod{2\pi}$ and $R_{ij} = \delta_{ij}$ is not acceptable since it does not give a diameter but leads to $\phi_1 = \phi_2 = \phi_3 = 0$, $\phi_4 = 1$.] Substituting back into Eq. (11) we get

$$\phi_1 = 2(\psi_2\psi_3 - \psi_1\psi_4)$$
$$\phi_2 = -2(\psi_1\psi_3 + \psi_2\psi_4)$$
$$\phi_4 = 1 - 2(\psi_1^2 + \psi_2^2)$$

By definition, this is the mapping $\varphi^{\#}$. The domain M^3 is seen to be a 3-sphere. After relabeling, $\phi_1 \to \xi_2$, $\phi_2 \to -\xi_1$, $\phi_4 \to \xi_3$, the above mapping is precisely the Hopf mapping of Eq. (6). The wave functional $\Psi(\varphi)$ can now be found according to Eq. (9) and Eq. (10).

6. DISCUSSION AND CONCLUSIONS

The wave function $\Psi_\pm(\varphi)$ of Section 4 can be compared to Skyrme's

$F_{\pm} = \exp(\pm i\alpha/2)$ (Skyrme, 1971; see also Skyrme, 1961b; Mandelstam, 1975; Aoyama and Kodama, 1976, and Aoyama, 1977). The quantity F is used in Skyrme's one-dimensional model and is a double-valued functional of $\alpha \in [0, 2\pi]$. It is an operator functional—Skyrme's approach being to look for field operators, as opposed to looking for wave functionals of field variables. Skyrme (1977) shows that F can be combined with a kink creation operator $K = \exp[2\pi i \int_{x_0}^{\infty} (\partial\alpha/\partial t)\, dx]$ to give a field operator satisfying Weyl's two-component spinor equation for neutrinos

$$(\pm \partial/\partial x - \partial/\partial t)(F_{\pm} K) = 0$$

A discussion of the fermionlike behavior of $F_{\pm} K$ is given by Ringwood (1979). F is the spinor part of the operator FK. In the same way, Ψ must be combined with some other functional to obtain an overall wave functional that could be expected to be the solution of some Schrödinger equation. The difficult problem of constructing such a quantity is clearly beyond the scope of this present work. We remark that Ragiadakos (1980) has recently studied a theory with solitons present and has found explicit solutions of the Schrödinger equation in the field representation.

Skyrme (1971) has also constructed operators analogous to F and K for his three-dimensional theory. The F operator (called S_1, in the three-dimensional theory) is $\exp(i\mathbf{t}\cdot\boldsymbol{\theta})$ where $\mathbf{t}$ is a constant vector and the $\{\theta_i\}$ are angular variables. The way that the $\{\theta_i\}$ are obtained as solutions of a certain nonlinear differential equation leads to double-valued spinorial behavior for F. Although F and $\Psi(\varphi)$ would appear to have a common topological origin, it seems difficult to establish their equivalence explicitly.

Other approaches to finding an underlying (topological) mechanism for half-odd-integer spin include Finkelstein (1955), Schulman (1968), Hellsten (1979, 1980a, 1980b), the study on monopoles by Ringwood and Woodward (1981), and the work on charge-monopole composites by Goldhaber (1976) and Leinaas (1978). For a general view of spin in the context of kink theory, see Finkelstein and Rubinstein (1968) and Finkelstein and Williams (1984). Even within the framework of the present paper a number of formalisms different from the one adopted could have been used. For example, the Hopf invariant could have been defined in terms of the linking number of two fibers in S^3 (Hilton, 1966, p. 70; Milnor, 1972, p. 54). If $\mathbf{r}_1$ and $\mathbf{r}_2$ denote position vectors locating the two fiber loops C_1 and C_2, the Hopf invariant is given by

$$L = (4\pi)^{-1} \oint_{C_1} \oint_{C_2} d\mathbf{r}_1 \times d\mathbf{r}_2 \cdot (\mathbf{r}_1 - \mathbf{r}_2) |\mathbf{r}_1 - \mathbf{r}_2|^{-3}$$

This linking number integral has been well known in physics since the time of Maxwell and has been used more recently in the work of Edwards (1968). Clearly, L must be equivalent to the Whitehead integral and an indefinite form of L could have been used instead of Eq. (9). Alternatively, we could have used a form of Kervaire integral, Eq. (7), with $n = 3$. The justification for replacing the usual $S^4 \to S^3$ spin formalism by $S^5 \to S^3$ comes from the isomorphism $\pi_4(S^3) \approx \pi_5(S^3)$.

In this paper we have constructed a spinor wave functional for a kink theory where the range of the field variables is a 3-sphere. Thus an object of half-odd-integer spin has been made from integer spin objects. Should one draw the conclusion that integer spin quantities are more basic than half-odd-integer spin quantities? This is certainly not the *simplest* approach to take, given that a spinor functional of spinless fields is such a complicated object [as exemplified by our functional $\Psi(\varphi)$ in Eqs. (9) and (10)] whereas spinless functions of spinor fields are so easily constructed (for example: $\phi = \bar{\psi}\psi$).

ACKNOWLEDGMENTS

The authors would like to thank C. S. Hoo for a number of useful comments and for helping with the proof of the proposition in Section 4. Discussions of the Whitehead integral with H. Schiff are gratefully acknowledged.

REFERENCES

Aoyama, S. (1977). Nagoya University Physics Department Preprint, DPNU-37-77.

Aoyama, S., and Kodama, Y. (1976). *Progress of Theoretical Physics*, **56**, 1970.

Conner, P. E., and Floyd, E. E. (1964). *Differentiable Periodic Maps*. Springer-Verlag, Berlin.

De Ritis, R., Finkelstein, D., Pisello, D. M., and Weil, D. (1978). *Il Nuovo Cimento*, **48A**, 184.

De Vega, H. J. (1978). *Physical Review D*, **18**, 2945.

Edwards, S. F. (1968). *Journal of Physics A*, **1**, 15.

Enz, U. (1963). *Physical Review*, **131**, 1392.

Finkelstein, D. (1955). *Physical Review*, **100**, 924.

Finkelstein, D. (1966). *Journal of Mathematical Physics*, **7**, 1218.

Finkelstein, D., and Misner, C. W. (1959). *Annals of Physics*, **6**, 230.

Finkelstein, D., and Rubinstein, J. (1968). *Journal of Mathematical Physics*, **9**, 1762.

Finkelstein, D., and Williams, J. G. (1984). *International Journal of Theoretical Physics*, to appear.

Flanders, H. (1963). *Differential Forms*. Academic Press, New York.

Goldhaber, A. S. (1976). *Physical Review Letters*, **36**, 1122.

Guillemin, V. W., and Golubitsky, M. (1973). *Singularities of Differentiable Mappings*. Springer-Verlag, Berlin.

Hellsten, H. (1979). *Journal of Mathematical Physics*, **20**, 2431.

Hellsten, H. (1980a). On the Visual Geometry of Spinors and Twistors, appeared in *Cosmology and Gravitation; Spin, Torsion and Supergravity*, NATO Advanced Study Institute Series: Series B, Physics; Volume 58, P. G. Bergmann and Venzo De Sabbata, eds. Plemun Press, New York.

Hellsten, H. (1980b). *A Visually Geometric Foundation of Spinor and Twistor Algebra*. Fysiska Institutionen, Stockholm, Sweden.

Hilton, P. J. (1966). *An Introduction to Homotopy Theory*. Cambridge University Press, London.

Hilton, P. J., and Wylie, S. (1967). *Homology Theory*. Cambridge University Press, London.

Husemoller, D. (1974). *Fiber Bundles*. Springer-Verlag, Berlin.

Kervaire, M. (1953). *Académie des Sciences (Paris), Comptes Rendus*, **237**, 1486.

Kimstedt, C. (1979). *Il Nuovo Cimento*, **53A**, 133.

Leinaas, J. M. (1978). *Il Nuovo Cimento*, **47A**, 19.

Mandelstam, S. (1975). *Physical Review D*, **11**, 3026.

Marx, E. (1969). *Il Nuovo Cimento*, **60A**, 683.

Milnor, J. W. (1972). *Topology from the Differentiable Viewpoint*. University Press of Virginia, Charlottesville, Virginia.

Minami, M. (1979). *Progress of Theoretical Physics*, **62**, 1128.

Minami, M. (1980), *Progress of Theoretical Physics*, **63**, 303.

Montgomery, D., and Zippin, L. (1955). *Topological Transformation Groups*. Interscience, New York.

Nicole, D. A. (1978). *Journal of Physics G*, **4**, 1363.

Pak, N. K., and Tze, H. C. (1979). *Annals of Physics*, **117**, 164.

Patani, A., Schlindwein, M., and Shafi, Q. (1976). *Journal of Physics A*, **9**, 1513.

Pisello, D. M. (1977). *International Journal of Theoretical Physics*, **16**, 863.

Pisello, D. M. (1978). *International Journal of Theoretical Physics*, **17**, 143.

Pisello, D. M. (1979). *Gravitation, Electromagnetism and Quantized Charge*. Ann Arbor Science Publishers, Ann Arbor, Michigan.

Ragiadakos, Ch. (1980). *Canadian Journal of Physics*, **58**, 554.

Ringwood, G. A. (1979). *Il Nuovo Cimento*, **54A**, 483.

Ringwood, G. A., and Woodward, L. M. (1981). *Physical Review Letters*, **47**, 625.

Ryder, L. H. (1980). *Journal of Physics A*, **13**, 437.

Schulman, L. (1968). *Physical Review*, **176**, 1558.

Skyrme, T. H. R. (1958). *Proceedings of the Royal Society of London*, **A247**, 260.

Skyrme, T. H. R. (1961a). *Proceedings of the Royal Society of London*, **A260**, 127.

Skyrme, T. H. R. (1961b). *Proceedings of the Royal Society of London*, **A262**, 237.

Skyrme, T. H. R. (1962). *Nuclear Physics*, **31**, 556.

Skyrme, T. H. R. (1971). *Journal of Mathematical Physics*, **12**, 1735.

Spanier, E. H. (1966). *Algebraic Topology*. McGraw-Hill, New York.

Steenrod, N. (1951). *Topology of Fiber Bundles*. Princeton University Press, Princeton, New Jersey.

Stong, R. E. (1968). *Notes on Cobordism Theory*. Princeton University Press, Princeton, New Jersey.

Whitehead, J. H. C. (1947). *Proceedings of the National Academy of Sciences USA*, **33**, 117.

Williams, J. G. (1970). *Journal of Mathematical Physics*, **11**, 2611.

Williams, J. G. (1979). *Canadian Journal of Physics*, **57**, 590.

Williams, J. G., and Zvengrowski, P. (1977). *International Journal of Theoretical Physics*, **16**, 755.

Woo, G. (1977). *Journal of Mathematical Physics*, **18**, 1756.